a) $y'' - 2y' + y = f(x) \qquad y(0) = y'(0) = 0$

CF is $y = Ae^x + Bxe^x = (A+Bx)e^x$

Try $y = [A(x) + B(x).x]e^x \Rightarrow y' = [\overbrace{A' + xB'}^{0} + B + A + xB]e^x$

& choose $A' + xB' = 0$ so $y'' = [B' + A' + B + xB' + B + A + xB]e^x$

& Substitution gives

$[(B' + \cancel{2B} + \cancel{A} + \cancel{xB}) - 2(\cancel{B} + \cancel{A} + \cancel{xB}) + (\cancel{A} + \cancel{xB})]e^x = f(x)$

$\Rightarrow B'e^x = f(x)$ & $B = b + \int_0^x e^{-u} f(u)\,du$

$A' = -xB' = -xe^{-x}f(x) \Rightarrow A = a - \int_0^x ue^{-u}f(u)\,du.$

Initial conditions need $y(0) = 0 \Rightarrow a = 0$ & $y'(0) = 0 \Rightarrow a + b = 0$

$\Rightarrow b = 0$ & $y(x) = xe^x \int_0^x e^{-u}f(u)\,du - e^x \int_0^x ue^{-u}f(u)\,du$

$= \int_0^x (x-u)\,e^{+(x-u)} f(u)\,du.$

b) Direct substitution of $y(x) = \int_C e^{xt} f(t)\,dt$ into

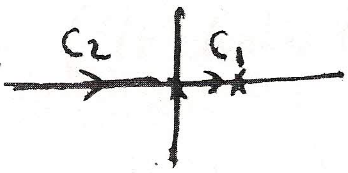
$xy'' + (n+1-x)y' - ny = 0$ gives $\int_C xe^{xt} f(t)(t^2 - t) + e^{xt} f(t)((n+1)t - n)\,dt = 0$

$\Rightarrow \left[e^{xt} f(t)(t)(t-1)\right]_C + \int_C e^{xt}\left[((n+1)t - n)f - \frac{d}{dt}[f(t^2 - t)]\right] dt = 0$

so choose f so that $[(n+1)t - n - 2t + 1]f = t(t-1)f'$

$\Rightarrow f'/f = \frac{(n-1)t - (n-1)}{t(t-1)} = \frac{n-1}{t} \Rightarrow f(t) = At^{n-1}$

$\Rightarrow y_i(x) = \int_C t^{n-1} e^{xt}\,dt$ where C_i is such that $\left[e^{xt} t^n (t-1)\right]_{C_i} = 0$



C_1 run from 0 to 1

C_2 run from $-\infty$ to 0

$$y_1(x) = \int_0^1 t^{n-1} e^{xt}\,dt \quad , \quad y_2(x) = \int_{-\infty}^0 t^{n-1} e^{+xt}\,dt = (-1)^{n+1}\int_0^{\infty} e^{-xt} t$$

As $x \to \infty$ $\quad y_1(x)$ is exponentially large, but $y_2(x) \to 0$

Hence $\quad y(x) = A\int_0^{\infty} e^{-xt} t^{n-1}\,dt = A\int_0^{\infty} e^{-u} \frac{u^{n-1}\,du}{x^n}$

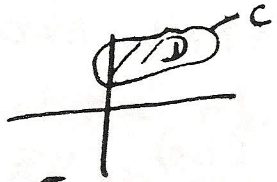
$$y(1) = A\int_0^{\infty} e^{-t} t^{n-1}\,dt \quad \text{so} \quad \underline{x^n y(x) = y(1)}$$

$t = 1-u$

$$\left. y_1(x) = \int_0^1 (1-u)^{n-1} e^{-xu}\,du \,.\, e^x \sim e^x/x \quad \text{as } x \to \infty \right\}$$

2 a) i) Phase plane is x-y plane & solutions $x(t), y(t)$ gives the parametric form of curves, trajectories, in the plane. A periodic solution has $x(t_0+T) = x(t_0)$ & $y(t_0+T) = y(t_0)$ for all t_0 & some T. These will be closed trajectories

ii) $$\int_D P_x + Q_y \, dx\,dy \overset{\text{Stoke's}}{=} \int_C P\,dy - Q\,dx$$



with D the interior of a closed trajectory, C

$$= \int_0^T P\underbrace{\frac{dy}{dt}}_{Q} - Q\underbrace{\frac{dx}{dt}}_{P}\,dt = \int_0^T PQ - QP\,dt$$

$$= 0$$

This is impossible if C lies in a region where $P_x + Q_y$ does not change sign.

b) i) $d^2x/dt^2 + dx/dt + x - x^3 = 0 \Rightarrow \frac{d^2x}{dt^2} = \frac{dy}{dt} = -\frac{dx}{dt} - x + x^3$

$= -y - x + x^3$ & $\frac{dx}{dt} = y$, as required.

ii) Here $P = dx/dt = y \Rightarrow P_x = 0$, $Q = dy/dt = -y - x + x^3 \Rightarrow$

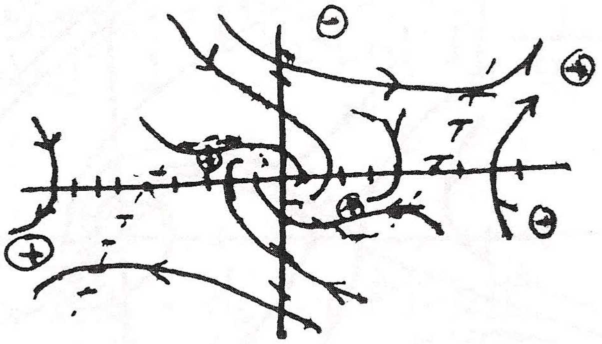
$Q_y = -1$ & so $P_x + Q_y = -1 < 0 \Rightarrow$ no periodic solutions

iii) Horiz null-lines have

$\frac{dy}{dt} = Q = 0 \Rightarrow y = -x + x^3$

Vert. null-clines have

$\frac{dx}{dt} = P = 0 \Rightarrow y = 0$



Trajectories have $\frac{dy}{dx} = \frac{Q}{P} = \frac{-y - x + x^3}{y}$

On $x = 0$ $dy/dx = -1$ & slope changes sign on crossing an null-cline

This allows something to be sketched, as shown

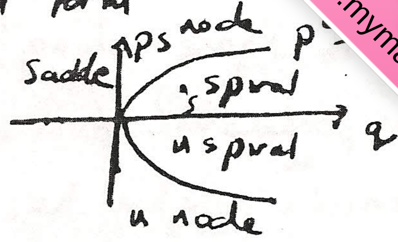
$\frac{dx}{dt}$ +ve $y > 0$, −ve $y < 0$ $\Rightarrow$ arrows

(It looks like $(\pm 1, 0)$ are saddle points
$(0,0)$ is a spiral (stable))

Critical points are where $P = Q = 0 \Rightarrow y = 0$, $x = -1, 0, 1$

Local to critical points, compare with standard form

$$\frac{dY}{dX} = \frac{CX+DY}{AX+BY}, \quad p = -(A+D), \quad q = AD - BC$$



Near $(0,0)$ $y \approx Y$, $x \approx X$ $\quad \frac{dY}{dX} = \frac{-X-Y}{Y}$

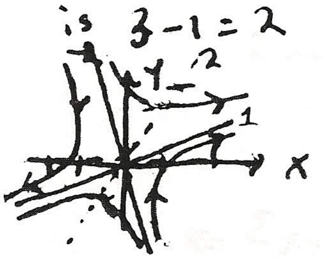
$\Rightarrow C = D = -1$, $A = 0$, $B = 1$ $\Rightarrow p = 1$, $q = 1$, $p^2 < 4q \Rightarrow$ stable spiral

Near $(1,0)$, $y \approx Y$, $x \approx 1 + X$, $\frac{dY}{dX} = \frac{2X-Y}{Y}$

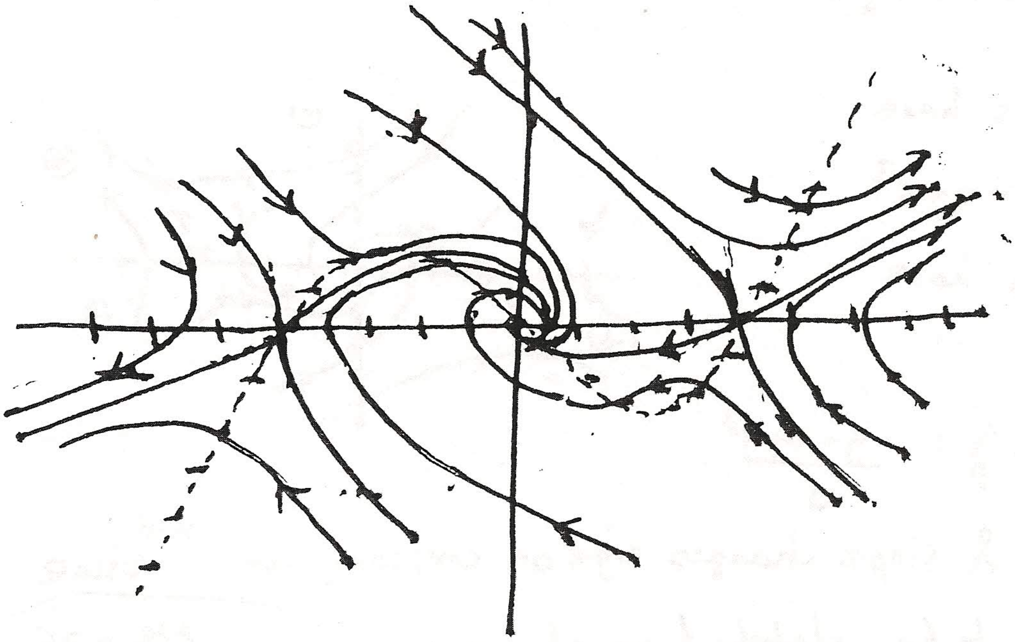
$\Rightarrow C = 2$, $D = -1$, $B = 1$, $A = 0 \Rightarrow p = 1$, $q = -2 < 0 \Rightarrow$ saddle

Lines $Y = mX$ through $X = Y = 0$ satisfy $m = \frac{2}{m} - 1$, $m^2 + m - 2 = 0$
$(m-1)(m+2) = 0$, $m = 1$ or $m = -2$

. Slope of horizontal nullcline $y = x^3 - x$ is $3 - 1 = 2$



The equations are left unchanged through $x \to -x$, $y \to -y$ so the pattern of trajectories in the 1st quadrant is mirrored in the 3rd & that in the second in the fourth.



) $\ddot{x} + \varepsilon(x^n - 1)\dot{x} + x = 0$

a) $\theta = (1 + \varepsilon n_1 + \ldots)t$, $(1+\varepsilon n_1 + \ldots)^2 x_{\theta\theta} + \varepsilon(x^n - 1)(1 + \varepsilon n_1 \ldots)x_\theta + \ldots$

& with $x = a\cos\theta + \varepsilon x_1(\theta) + \ldots$, we require, at $O(\varepsilon)$, $x_0 = a\cos\theta$,

$$x_{1\theta\theta} + 2n_1 x_{0\theta\theta} + (x_0^n - 1)x_{0\theta} + x_1 = 0 \qquad \text{(this method seen before)}$$

$$\Rightarrow x_{1\theta\theta} + x_1 = 2n_1 a\cos\theta + a\sin\theta(a^n\cos^n\theta - 1)$$

We ensure that x_1 is bounded by making sure that the r.h.s has no component proportional to $\cos\theta$ & to $\sin\theta$. We can set the Fourier cosine & sine components of r.h.s to zero, to obtain

$$2n_1 a \cdot 2\pi \cdot \frac{1}{2} + a\underbrace{\int_0^{2\pi} \sin\theta(a^n\cos^n\theta - 1)\cos\theta\, d\theta}_{=0 \text{ as } \sin\theta \text{ is odd about } \theta=\pi,\ \cos\theta \text{ is even.}} = 0 \Rightarrow n_1 = 0$$

$$\& \quad 0 + a\int_0^{2\pi} \sin^2\theta(a^n\cos^n\theta - 1)\, d\theta = 0$$

$$\Rightarrow a^n = I/J \text{ or } a = 0, \quad I = \int_0^{2\pi} \sin^2\theta\, d\theta, \quad J = \int_0^{2\pi} \sin^2\theta\cos^n\theta\, d\theta$$

$$I = 2\pi \cdot \frac{1}{2} = \pi, \quad J = \int_0^{2\pi} \cos^n\theta\, d\theta - \int_0^{2\pi} \cos^{n+2}\theta\, d\theta.$$

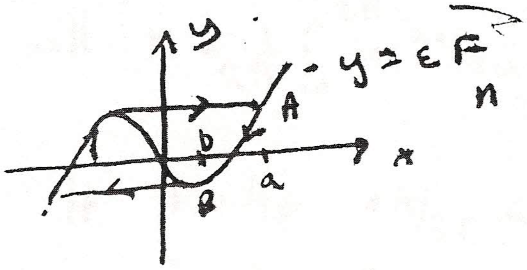
If n is odd, $J = 0$ & we are led to $a = 0$ & a contradiction & no such solution. If n is even we have

$$J = 2\pi\left[\frac{n!}{2^n[(n/2)!]^2}\right]\left[1 - \frac{(n+2)(n+1)}{4[((n+2)/2)]^2}\right] = 2\pi\left[\frac{n!}{2^n[(n/2)!]^2}\right]\left[\frac{1}{n+2}\right]$$

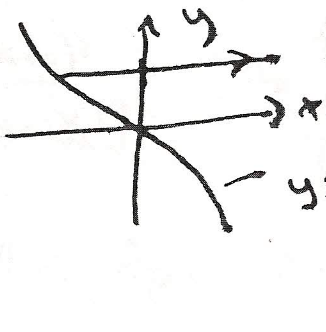
$$\& \quad a^n = 2^{n-1}(n+2)[(n/2)!]^2/n!$$

b) We have $dx/dt = y - \varepsilon F$ & $\dfrac{d^2x}{dt^2} = \dfrac{dy}{dt} - \varepsilon F'(x)\dfrac{dx}{dt}$

$$= -x - dx/dt \cdot (x^n - 1)\varepsilon \Rightarrow \ddot{x} + \varepsilon(x^n - 1)\dot{x} + x = 0 \qquad \left(F(x) = \frac{x^{n+1}}{n+1} - x\right)$$



n even



$y = \varepsilon F$, n odd

If $\varepsilon \gg 1$ & y not close to εF, then $dx/dt = O(\varepsilon) \gg 1$. So x increases rapidly, but as x must remain finite, it cannot do so for long. Hence as x increases rapidly, y remains approximately constant as $dy/dt = -x$, which is finite.

So trajectories are approximately $y = \text{const}$. However $\dot{x}$ & $\dot{y}$ are ~~or~~ not large if $y = \varepsilon F$. So closed trajectories are possible, as shown in ①, but not possible in ② where n is odd. (seen for $n=2$)

Period is dominated by slower motion along $y = \varepsilon F$ & from symmetry it is $$2\int_A^B dt = 2\int_a^b \frac{dt}{dx}\,dx = 2\int_a^b \frac{dt}{dy}\cdot\frac{dy}{dx}\,dx$$

$$= 2\int_a^b \varepsilon \frac{F'(x)}{-x}\,dx = \varepsilon 2\int_b^a \frac{x^n - 1}{x}\,dx$$

n even $\Rightarrow (n+1)$ odd.

Minimum at B has $F' = 0$ & $x = b = 1$. $F(-1) = \frac{(-1)^{n+1}}{n+1} + 1 = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

a is root of $F(\alpha) = \frac{+n}{n+1} \Rightarrow \alpha^{n+1} - (n+1)\alpha - n = 0$

4) a) $\ddot{x} + x = \varepsilon f(\dot{x})$. Introduce $T = \varepsilon t$ & treat t & T as independant variables & write $x = x_0(t,T) + \varepsilon x_1(t,T) + \ldots$

$$x_{0tt} + 2\varepsilon \frac{\partial}{\partial T} x_{0t} + \ldots + \varepsilon x_{1tt} + \ldots + (x_0 + \varepsilon x_1 \ldots) = \varepsilon f(x_{0t} + \ldots)$$

$$\Rightarrow \quad x_{0tt} + x_0 = 0 \quad \& \quad x_0 = A(T)\sin(t + \varphi(T))$$

$$x_{1tt} + x_1 = f(A\cos(t+\varphi)) - 2\frac{\partial}{\partial T}(A\cos(t + \varphi(T)))$$

Keeping x_1 bounded needs Fourier sine & cosine component of r.h.s $= 0$. $\frac{\partial}{\partial T} A\cos(t+\varphi) = \frac{\partial A}{\partial T}\cos\chi - A\frac{\partial \varphi}{\partial T}\sin\chi$, $\chi = t+\varphi$

So $2\pi \cdot \frac{1}{2} \cdot 2\frac{\partial A}{\partial T} = \int_0^{2\pi} \cos\chi \, f(A\cos\chi)\, d\chi.$

& $2\pi \cdot \frac{1}{2} \cdot 2A\frac{\partial \varphi}{\partial T} = -\int_0^{2\pi} \sin\chi \, f(A\cos\chi)\, d\chi.$ } as required

b) $\int_0^{2\pi} \sin\chi \, f(A\cos\chi)\, d\chi = 0$ as $\cos\chi$ is even & $\sin\chi$ odd about $x = \pi$.

$$\Rightarrow \frac{\partial \varphi}{\partial T} = 0, \quad \varphi = \varphi_0 \text{ a constant}$$

c) $$\frac{\partial A}{\partial T} = \frac{1}{2\pi}\int_0^{2\pi} A\cos^2\chi - \alpha A^n \cos^{n+1}\chi \, d\chi$$

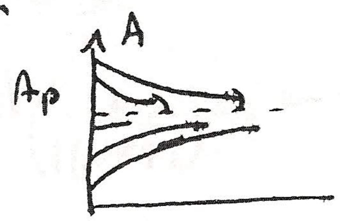
If n is even, $n+1$ is odd & $\int_0^{2\pi} \cos^{n+1}\chi \, d\chi = 0$

$$\Rightarrow \frac{dA}{dT} = \frac{1}{2\pi}\cdot A \cdot \frac{2\pi}{2} = \frac{A}{2} \Rightarrow A \text{ grows exponentially}$$

d) If n is odd, $n+1$ is even

$$\frac{\partial A}{\partial T} = \frac{1}{2\pi}\int_0^{2\pi} A\cos^2\chi - \alpha A^n \cos^{n+1}\chi \, d\chi = \frac{A}{2} - \alpha A^n K$$

with $K = \frac{1}{2\pi}\int_0^{2\pi} \cos^{n+1}\chi \, d\chi = \frac{(n+1)!}{2^{n+1}[((n+1)/2)!]^2}$



If $A > A_p$ $\partial A/\partial T < 0$
$A < A_p$ $\partial A/\partial T > 0$ } $\Rightarrow A \to A_p$ as $T \to \infty$ $\quad A_p = \left(\frac{1}{2\alpha K}\right)^{n-1}$

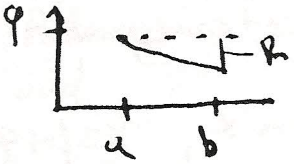
5 a) i $\int_0^1 e^{-xt} f(t)\,dt \sim \dfrac{a_0}{x^{\lambda_0+1}}\lambda_0! + \dfrac{a_1\lambda_1!}{x^{\lambda_1+1}} + \dfrac{a_2\lambda_2!}{x^{\lambda_2+1}}$

where, if $T=\infty$, $f(t)$ does not grow superexponentially as $t\to\infty$

and in the vicinity of $t=0$, $f(t) \sim a_0 t^{\lambda_0} + a_1 t^{\lambda_1} + a_2 t^{\lambda_2} + \dots$

$\lambda_0 < \lambda_1 < \lambda_2 \dots$ ($\lambda_0 < -1$ so $\int$ exists)

ii) Using Watson's Lemma, write $u = \varphi(a) - \varphi(t)$ & $R = \varphi(a) - \varphi(b)$



$$\int_a^b e^{x\varphi(t)} f(t)\,dt = \int_0^R e^{x\varphi(a)} e^{-xu} f(t) \frac{du}{-\varphi'(t)}$$

Near $t=a$, $u=0$ $f/\varphi' \approx f(a)/\varphi'(a)$ & Watson's Lemma gives

$$\underline{\frac{e^{x\varphi(a)}}{x} \cdot \frac{f(a)}{-\varphi'(a)}} \qquad \text{(seen)}$$

If max is at $x=b$, result is $\underline{\dfrac{e^{x\varphi(b)} f(b)}{x\,\varphi'(b)}}$

Alternatively $\int_a^b e^{x\varphi(t)} f(t)\,dt = \frac{1}{x}\int_a^b x\varphi'(t) e^{x\varphi(t)} \frac{f(t)}{\varphi'(t)}\,dt$

$$= \left[\frac{f(t)}{x\varphi'(t)} e^{x\varphi(t)}\right]_a^b - \frac{1}{x}\int_a^b e^{x\varphi(t)} \frac{d}{dt}\left[\frac{f(t)}{\varphi'(t)}\right] dt \qquad \text{(seen)}$$

second term, repeating argument, is $O(1/x)$ smaller.

In first term take the biggest of $e^{x\varphi(b)}$ or $e^{x\varphi(a)}$

iii) Using Watsons lemma, proceed as above but near $t=a$, $u=0$, write $u = \varphi(a) - \varphi(a) - (t-a)\varphi'(a) - \frac{1}{2}(t-a)^2\varphi''(a) \dots$

$$= \frac{(t-a)^2}{2}|\varphi''(a)| \qquad (\varphi''(a) < 0 \text{ as we have max})$$

$\Rightarrow (t-a) = \left(\dfrac{2u}{|\varphi''(a)|}\right)^{1/2}$. $\varphi'(t) = \underset{0}{\varphi'(a)} + (t-a)\varphi''(a) + \dots$

$$= \left(\frac{2u}{|\varphi''(a)|}\right)^{1/2}(-1)|\varphi''(a)|$$

& in the vicinity of $u=0,\ t=a$ $\quad \dfrac{f(t)}{-\varphi'(t)} \approx \dfrac{f(a)}{(2|\varphi''(a)|\cancel{u})^{1/2}u}$

& Watsons Lemma gives

$$\sim e^{x\varphi(a)} \frac{f(a)}{(2|\varphi''(a)|)^{1/2}} \cdot \frac{(-1/2)!}{x^{1-1/2}} = e^{x\varphi(a)} \frac{f(a)}{2} \sqrt{\frac{2\pi}{|\varphi''(a)|x}}$$

$(-1/2)! = \sqrt{\pi}$

or, expanding close to maximum in φ,

$$\int_a^b e^{x\varphi(t)} f(t)\,dt \sim \int_a^b e^{x\varphi(a) + x\frac{\varphi''(a)}{2}(t-a)^2 + \dots} f(a)\,dt$$

$$\approx \int_0^\infty e^{-x\frac{|\varphi''(a)|}{2}u^2} \cdot f(a)\,du \cdot e^{x\varphi(a)} = e^{x\varphi(a)} \cdot f(a) \cdot \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{x|\varphi''(a)|/2}}$$

(both seen)

b) i) Using Watson's Lemma

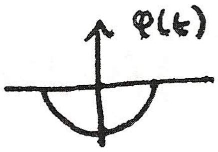
$$\int_0^\infty e^{-xt^2} \sin t\,dt = \int_0^\infty e^{-xu} \frac{\sin\sqrt{u}}{2\sqrt{u}}\,du \quad \& \quad \frac{\sin\sqrt{u}}{\sqrt{u}} \sim 1 \text{ as } u \to 0$$

$u = t^2$
$du = 2t\,dt$

$a_0 = 1/2,\ \lambda_0 = 0$

$$\sim \frac{1}{2x}$$

ii) $\displaystyle\int_{-\pi/2}^{\pi/2} (t+2)\, e^{-x\cos t}\,dt$



$\varphi'(t) = \sin t$
$= -1$ at $t = -\pi/2$
$= 1$ at $t = \pi/2$

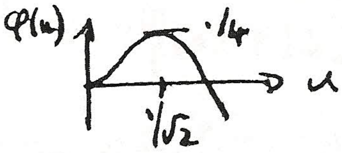
$\sim$ two contributions, one from each interval $[-\pi/2, 0]$ & $[0, \pi/2]$ $\quad \varphi(-\pi/2) = \varphi(\pi/2) = 0$

$$\sim -e^{x \cdot 0} \frac{(-\pi/2 + 2)}{x(-1)} + e^{x \cdot 0} \frac{(\pi/2 + 2)}{x \cdot 1} = \frac{4}{x}$$

iii) Put $t = xu$ (this not seen, but suggested by form of question ($g(t/x)$))

$$\int_0^\infty e^{x^2t^2-t^4}\, g(t/x)\,dt = \int_0^\infty e^{x^4(u^2-u^4)}\, g(u)\, x\,du$$

$$= x\int_0^\infty e^{x^4\varphi(u)}\, g(u)\,du$$



Max in $\varphi(u)$ is where $2u = 4u^3 \Rightarrow u = 1/\sqrt{2}$, $\varphi(u) = 1/2 - 1/4 = 1/4$

$\varphi''(u) = 2 - 12u^2 = 2 - 6 = -4$ at $1/\sqrt{2}$

Answer is half of that given in (a) (iii) as max is in the middle of range of integration. (argument seen)

$$\int_0^\infty e^{x^2t^2-t^4}\, g(t/x)\,dt \approx 2x\,\frac{1}{2}\, e^{x^4 \cdot 1/4} \cdot g(1/\sqrt{2}) \sqrt{\frac{2\pi}{(x^4)^{1/2}\cdot 4}}$$

$$\sim \frac{e^{\frac{1}{4}x^4}}{x} \cdot \sqrt{\frac{\pi}{2}} \cdot g(1/\sqrt{2})$$